A Tractable Approximation for Stochastic MPC and Application to Mechanical Pulping Processes

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Abstract

This paper develops a tractable approximation for stochastic model predictive control (SMPC). Under the proposed approach, we solve multiple deterministic MPC (DMPC) problems over individual scenarios of the uncertain variables to obtain a set of control policies and select from this candidate set a control input that yields the best approximation of the SMPC solution (i.e., yields the smallest statistical measure of the objective function (e.g., expected value) and of the constraints). This approach is a scenario decomposition scheme that overcomes tractability issues of SMPC (which solves problems that incorporate multiple scenarios all-at-once). Moreover, the approach enables flexible handling of complex statistical measures (e.g., medians, quantiles, and chance constraints) and enables prioritization of objectives and constraints (this is difficult to do with off-the-shelf optimization solvers). An application to a nonlinear mechanical pulping process demonstrates that the approach provides high quality solutions. We hypothesize that this is because the optimal SMPC policy lives in a space that is spanned by the control policies for the individual scenarios. Moreover, we note that a traditional DMPC policy corresponds to the policy of an individual scenario (the mean scenario is typically chosen). Consequently, the proposed approach can do no worse than DMPC and can be interpreted as an approach that seeks to find a DMPC policy that best approximates the SMPC policy.

Keywords: Stochastic model predictive control; economics; pulp and paper

1. Introduction

Model predictive control (MPC) has been widely studied in academia and adopted in industry as an effective strategy to deal with multivariable constrained control problems [1, 2, 3]. MPC aims at determining a closed-loop control policy by recursively solving an open-loop finite horizon problem. Deterministic MPC (DMPC) does not provide a systematic approach to capture model uncertainties and disturbances in the computation of control policies and this can lead to poor performance and constraint violations [4]. In

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contrast, stochastic MPC (SMPC) factors in uncertainty information directly in the control formulation [5, 6, 7]. This approach leads to improved constraint satisfaction and performance and has recently seen applications in building climate control, wind turbines, batteries, power generation and distribution, and network traffic control [5, 6, 8, 9, 10, 11, 12].

A major issue associated with SMPC is *computational tractability* [5, 13, 14]. Tractability issues arise because the control formulation incorporates statistical measures for the objective and constraints (which are random variables). For instance, in the most basic SMPC formulation, one seeks to minimize the expected value of the objective function (this involves a high-dimensional integral). As a result, one needs to approximate the expectation using quadrature techniques such as Monte Carlo sampling or polynomial chaos expansions [15]. These *transcription* approaches convert the infinite-dimensional SMPC problem into a standard (finite-dimensional) optimization problem that can be handled using off-the-shelf solvers. Unfortunately, the resulting optimization problems are often computationally expensive (e.g., they may require many scenarios to approximate statistical measures). These computational tractability issues are exacerbated in more sophisticated SMPC formulations in which one might seek to optimize complex statistical measures (e.g., variance, conditional value at risk, quantiles, medians). Under such formulations, as the resulting optimization problems might involve integer variables, complex nonlinearities, or even bilevel formulations. Similar tractability issues arise when dealing with constraints; specifically, constraints for SMPC are often enforced using statistical measures such as chance constraints, quantiles, risk measures, and almost-surely constraints (i.e., constraints are satisfied for all scenarios) [5].

In this work, we explore tractable approximations for SMPC. Under the proposed paradigm, we use sample scenarios to transform statistical measures for the objective and constraints into finite-dimensional representations [5, 16, 17]. To deal with tractability issues of the resulting optimization problems, we propose an approximation technique that is inspired by the quantile scenario analysis method proposed in [9]. In this approach, we solve multiple DMPC problems for different scenarios to obtain a set of candidate control policies. The observation is that these policies can be computed quickly and in parallel as they do not involve statistical measures. The set of computed control policies forms a candidate set from which we select the policy that best approximates the SMPC solution (i.e., yields the smallest statistical measures for the objectives and constraints). This approach allows us to handle complex measures and allows us to prioritize conflicting objectives such as economics and stability. The proposed approach only provides an approximate policy of the SMPC problem but we note that this approach can be interpreted as a controller that seeks to find the DMPC policy that *best approximates* the performance of SMPC. Moreover, since DMPC policy is equivalent to solving a problem with one scenario (typically the mean or worst case), the proposed approach can do no worse than the standard DMPC policy. We demonstrate the developments using a stochastic version of economic MPC [18] applied to a mechanical pulping (MP) process. In this process, we seek to drive transitions between steady-states that deliver product with desired characteristics and while minimizing energy consumption [19, 20, 21]. The process is challenging in that it involves multiple sources of uncertainty and strong nonlinearities. The proposed framework extends recent work in deterministic economic MPC for MP processes [22, 23].

The paper is structured as follows. In Section 2, we provide a discussion of the SMPC problem under study and Section 3 presents the proposed approximation strategy. Section 4 presents an MP process to demonstrate the performance of the proposed approach followed by conclusions in Section 5.

2. Stochastic Model Predictive Control Formulation

In this section, we frame the stochastic model predictive control (SMPC) formulation under study. We let $\zeta_t^{(S)} := \{\zeta_{0|t}^1, \ldots, \zeta_{N-1|t}^S\}$ and $\eta_t^{(S)} := \{\eta_{1|t}^1, \ldots, \eta_{N|t}^S\}$ denote *i.i.d.* samples (drawn at time t) for the random variables ζ_t and η_t, respectively. We denote the prediction horizon of the controller as N. The SMPC formulation has the form:

$$\min_{v_{0|t},\cdots,v_{N-1|t}} \quad \frac{1}{S}\sum_{i=1}^{S}\sum_{k=0}^{N-1} L^{ec}(z_{k|t}^i, v_{k|t}), \tag{1a}$$

$$\text{s.t.} \quad z_{k+1|t}^i = f(z_{k|t}^i, v_{k|t}, \zeta_{k|t}^i), \quad k = 0, \cdots, N-1, \tag{1b}$$

$$y_{k|t}^i = g(z_{k|t}^i) + \eta_{k|t}^i, \quad k = 1, \cdots, N, \tag{1c}$$

$$z_{0|t}^i = x_t, \quad z_{N|t}^i \in \mathbb{X}_f, \quad z_{k|t}^i \in \mathbb{X}, \quad k = 1, \cdots, N-1, \tag{1d}$$

$$v_{k|t} \in \mathbb{U}, \quad k = 0, \cdots, N-1, \tag{1e}$$

$$\sum_{k=0}^{N-1} L^{tr}(z_{k|t}^i, v_{k|t}) \leq \epsilon_t, \quad k = 0, \ldots, N-1. \tag{1f}$$

where the stage cost is given by $L^{ec}(z_{k|t}^i, v_{k|t})$ and $z_{k|t}^i$, $y_{k|t}^i$ denote the k-step-ahead predictions of state and output variables at time t and for the i-th scenario. With the scenario set $\{\zeta_t^{(S)}, \eta_t^{(S)}\}$, the system dynamics (1b) provide S different state trajectories over the prediction horizon, each corresponding to a particular scenario $\{\zeta_{k|t}^i, \eta_{k|t}^i\}_{k=0}^{N-1}$. The objective function is the expected value cost, which is approximated using a sample average with S scenarios. The state measurement at the current sampling time t (x_t) is used as initial state. For convenience, we define the economic value function evaluated at scenario i and under input $v_{k|t}$ as:

$$V_t^{ec}(v_{k|t}, \{\zeta_{k|t}^i, \eta_{k|t}^i\}) \quad := \quad \sum_{k=0}^{N-1} L^{ec}(z_{k|t}^i, v_{k|t}) \tag{2}$$

The SMPC problem seeks to find the optimal policy $\{\bar{v}_{0|t}, \cdots, \bar{v}_{N-1|t}\}$ that minimizes the expected cost and satisfies constraints (1a)–(1f). Only the first element of the policy $u_t := \bar{v}_{0|t}$ is injected into the plant. The constraints of the SMPC formulation include standard input and state constraints and a stabilizing constraint. To construct the stabilizing constraint, we assume that the stage cost L^{tr} is a positive definite function (e.g., as in a tracking problem) and consider a sequence $\{\epsilon_t\}$ that decreases as $t \to \infty$. Details on how to construct such a sequence can be found in [24, 18] (details are omitted here for brevity). For convenience, we also define the tracking value function at scenario i and input $v_{k|t}$ as:

$$V_t^{tr}(v_{k|t}, \{\zeta_{k|t}^i, \eta_{k|t}^i\}) \quad := \quad \sum_{k=0}^{N-1} L^{tr}(z_{k|t}^i, v_{k|t}) \tag{3}$$

We thus see that the stabilizing constraint seeks to progressively decrease in the tracking function in order to ensure stability [23, 18, 25]. This constraint is necessary because minimization of the economic cost function does not guarantee stability [26]. In our previous work, we developed a deterministic variant of this economic MPC formulation [22, 23]. The formulation considered here is a stochastic variant of such formulation.

In summary, the idea behind the SMPC controller is to compute an optimal control policy that minimizes the expected economic cost (corresponding to using the expected value as the statistical measure). An issue with the use of the expected value as a measure is that it might lead to poor performance in extreme scenarios. The proposed SMPC formulation can thus be modified to by using alternative statistical measures such as the quantile or the median (quantile at a probability of 50%) of the economic cost (this is a more robust approach to deal with extreme scenarios). The control policy computed with SMPC must also be feasible under all S scenarios. In other words, the SMPC formulation enforces satisfaction of state constraints and decrease of the tracking value function for all scenarios S. We note that assuming that the constraints hold for all scenarios is equivalent to say that they hold with probability one:

$$\mathbb{P}\left(\sum_{k=0}^{N-1} L^{tr}(z_{k|t}, v_{k|t}) \leq \epsilon_t\right) = 1. \tag{4}$$

Consequently, this formulation might be restrictive. The proposed SMPC formulation can thus be modified to enforce constraints by using alternative measures such as a chance constraint in which the constraints are enforced with a probability lower than one. In addition, we can relax the satisfaction of the stabilizing constraint for all scenarios by requiring satisfaction of a probability $\rho < 1$:

$$\mathbb{P}\left(\sum_{k=0}^{N-1} L^{tr}(z_{k|t}, v_{k|t}) \leq \epsilon_t\right) \geq \rho \tag{5}$$

or by requiring satisfaction in expectation:

$$\frac{1}{S}\sum_{i=1}^{S}\sum_{k=0}^{N-1} L^{tr}(z_{k|t}^i, v_{k|t}) \leq \epsilon_t \tag{6}$$

While the SMPC formulation makes practical sense, it can be challenging to solve in real-time. This can be due to the need to handle many scenarios and/or from the need to capture complex measures (e.g., quantiles and chance constraints). Consequently, we are interested in developing strategies that compute approximate control policy.

3. Approximating the Policy of Stochastic MPC

In this paper, we propose a strategy to compute an approximate policy for SMPC. For each scenario $i = 1, ..., S$, an optimal policy $\{\bar{v}^i_{0|t}, \dots, \bar{v}^i_{N-1|t}\}$ is computed by solving a DMPC problem of the form:

$$\min_{v^i_{0|t},\cdots,v_{N-1|t}} \quad \sum_{k=0}^{N-1} L^{ec}(z_{k|t}, v_{k|t}), \tag{7a}$$

$$s.t. \quad z_{0|t} = x_t, \quad z_{N|t} \in \mathbb{X}_f, \tag{7b}$$

$$z^i_{k+1|t} = f(z_{k|t}, v_{k|t}, \zeta_{k|t}), \quad k = 0, \dots, N-1, \tag{7c}$$

$$y^i_{k|t} = g(z^i_{k|t}) + \eta_{k|t}, \quad k = 1, \dots, N, \tag{7d}$$

$$z^i_{k|t} \in \mathbb{X}, \quad v_{k|t} \in \mathbb{U}, \quad k = 0, \dots, N-1, \tag{7e}$$

$$\sum_{k=0}^{N-1} L^{tr}(z_{k|t+1}, v_{k|t+1}) \leq \epsilon_t, \quad k = 0, \dots, N-1, \tag{7f}$$

The corresponding value function for the i-th scenario is given by $V^{ec}_t(\bar{u}^i_{0|t}, \{\zeta^i_{k|t}, \eta^i_{k|t}\})$. We define the policy candidate set over the S scenarios as $\mathbf{\bar{u}_0} = \{\bar{u}^1_{0|t}, \cdots, \bar{u}^S_{0|t}\}$.

Given the optimal policy $\bar{u}^i_{0|t}$ for the i-th scenario, we evaluate the value functions $V^{ec}_t(\bar{u}^i_{0|t}, \{\zeta^j_{k|t}, \eta^j_{k|t}\})$ using this policy over for the full set of scenarios $j \neq i$. Our goal now is to select a *control policy* from the candidate set to be implemented in the system. This selected policy must solve (or approximately solve) the SMPC problem. In the context of problem (1), we want a control policy that minimizes the expected cost and satisfies the state and stabilizing constraints for all scenarios (input constraints are satisfied by construction).

To select a policy, we construct a coordinating matrix for the cost (denoted as V^{ec}_{set} and shown in (8)); here; each row corresponds to a candidate control (in the left row outside of the matrix) and each column to a scenario (top row outside of the matrix). We note that the diagonal elements correspond to the optimal cost of problem (7) for all scenarios i.

$$V^{ec}_{set} := \begin{array}{c} \\ \bar{u}^1_{0|t} \\ \bar{u}^2_{0|t} \\ \vdots \\ \bar{u}^S_{0|t} \end{array} \begin{array}{c} \begin{array}{cccc} \{\zeta^1_{k|t}, \eta^1_{k|t}\} & \{\zeta^2_{k|t}, \eta^2_{k|t}\} & \cdots & \{\zeta^S_{k|t}, \eta^S_{k|t}\} \end{array} \\ \begin{pmatrix} \bar{V}^{ec}_t(\bar{u}^1_{0|t}, \{\zeta^1_{k|t}, \eta^1_{k|t}\}) & V^{ec}_t(\bar{u}^1_{0|t}, \{\zeta^2_{k|t}, \eta^2_{k|t}\}) & \cdots & V^{ec}_t(\bar{u}^1_{0|t}, \{\zeta^S_{k|t}, \eta^S_{k|t}\}) \\ V^{ec}_t(\bar{u}^2_{0|t}, \{\zeta^1_{k|t}, \eta^1_{k|t}\}) & \bar{V}^{ec}_t(\bar{u}^2_{0|t}, \{\zeta^2_{k|t}, \eta^2_{k|t}\}) & \cdots & V^{ec}_t(\bar{u}^2_{0|t}, \{\zeta^S_{k|t}, \eta^S_{k|t}\}) \\ \vdots & \vdots & \ddots & \vdots \\ V^{ec}_t(\bar{u}^S_{0|t}, \{\zeta^1_{k|t}, \eta^1_{k|t}\}) & V^{ec}_t(\bar{u}^S_{0|t}, \{\zeta^2_{k|t}, \eta^2_{k|t}\})) & \cdots & \bar{V}^{ec}_t(\bar{u}^S_{0|t}, \{\zeta^S_{k|t}, \eta^S_{k|t}\}) \end{pmatrix} \end{array} \tag{8}$$

In an *ideal case*, the state and stabilizing constraints are satisfied for all scenarios $j \neq i$ and for all candidate inputs $\bar{u}^i_{0|t}$. In such a case, our strategy simply selects the input that leads to the smallest expected cost (column-wise average). This approximation approach is fast because it only requires solving decoupled scenario problems (which can be done individually and in parallel). Notably, this approach can also be used to compute actions that minimize alternative statistical measures for the cost. For instance, we can also select the control candidate that achieves the smallest quantile of the cost [9]. This can be done trivially using information from the cost matrix but doing so directly in an SMPC formulation is non-trivial (quantiles do not have a simple algebraic forms as in the case of expected values). Our approach thus seeks to not only find approximate policies faster but also to enable the use of alternative statistical measures in SMPC.

The proposed approximation approach presents interesting properties. Specifically, we observe that the sample average of the diagonal entries in V_{set}^{ec} provides a lower bound for the optimal cost of SMPC. In the stochastic programming literature, this lower bound is the so-called wait-and-see cost [27]. Consequently, it is possible to estimate an optimality gap for every candidate control (estimate how far is the approximate policy from the actual SMPC policy).

We also highlight that the candidate controls considered are not constructed arbitrarily but are built by exploring the actual uncertainty space. As a result, we expected that the optimal SMPC policy is in a space spanned by the candidate controls (or at least close to that space). In fact, we highlight that some of the candidate control policies correspond to policies computed under typical DMPC formulations. Specifically, in a DMPC formulation, one often selects a single representative scenario ($S = 1$) for the random variables (typically the mean or the worst-case) to compute the control. Consequently, we observe that the policy obtained with the proposed approximation approach *can do no worse than a typical DMPC policy* in terms of variance. Our approach can also be interpreted as a strategy that seeks to improve the DMPC policy or as a strategy that seeks to find a deterministic policy that best approximates the SMPC policy. The detailed approximation scheme is outlined in Table 1.

Table 1: Implementation of proposed approximate SMPC scheme in ideal case

Algorithm for the ideal case

Input: $x_0 \in \mathbb{X}$, $\sigma \in [0, 1)$, set $t \leftarrow 0$ and $\epsilon_0 \leftarrow +\infty$.

For $t = 0, \ldots,$ simulation duration **do**

1: Draw S scenarios of $\{\zeta_{k|t}^i, \eta_{k|t}^i\}_{k=0}^{N-1}$, $i = 1, \ldots, S$.

2: **for** $i = 1, \ldots, S$ **do**

2.1. Compute optimal input $\bar{u}_{0|t}^i$ and its corresponding optimal value function $\bar{V}_t^{ec}(\bar{u}_{0|t}^i, \{\zeta_{k|t}^i, \eta_{k|t}^i\})$ for the i-th scenario.

2.2. Given $\bar{u}_{0|t}^i$ computed in step 2.1, evaluate $V_t^{ec}(\bar{u}_{0|t}^i, \{\zeta_{k|t}^j, \eta_{k|t}^j\})$ for the rest of scenarios $\{\zeta_{k|t}^j, \eta_{k|t}^j\}_{k=0}^{N-1}$, $\forall j = 1, \cdots, S$, and $j \neq i$.

end for

3: Construct matrix V_{set}^{ec} as structured in (8).

4: Find the minimal measure in each row of V_{set}^{ec} and its corresponding input $\bar{u}_{0|t}^l$, $l \in 1, \cdots, S$. Set $u_t \leftarrow \bar{u}_{0|t}^l$.

5: Implement u_t to the plant and obtain the state variables x_{t+1}.

6: Set $\epsilon_{t+1} < \epsilon_t$.

End

In the ideal case, we assume that the state and stabilizing constraints hold for all scenarios and all candidate controls. In other words, it is assumed that a single input can satisfy all constraints. However, in practice, we expect this assumption not to hold. In fact, it is possible that *the SMPC formulation does not even have a feasible solution*. Because of this, one must select a control that allows for some constraint violations. One possibility to deal with this is the following: given scenarios $\{\zeta^j_{k|t}, \eta^j_{k|t}\}_{k=0}^{N-1}$, $j = 1, \ldots, S$, and candidate control policies, we construct matrix V^{ec}_{set} as in the ideal case. For a given candidate policy $\bar{u}^i_{0|t}$, $i \in \{1, \cdots, S\}$, we count the number of state constraint (7e) violations (denoted as M_i) and count the number of stabilizing constraint (7f) violations (denoted as N_i) over all scenarios $j = 1, \ldots, S$. We then single-out policies $\bar{u}^i_{0|t} \in \bar{\mathbf{u}}_\mathbf{0}$ that have the smallest faction of violations N_i/S, $i \in \{1, \ldots, S\}$ and that satisfy $M_i/S \leq \rho$ (where $\rho \in [0, 1]$ is a given probability level). By construction, we have that $N_i \leq S$ and $M_i \leq S$ and we thus note that minimizing N_i/S is equivalent to minimizing the probability (frequency) of state constraint violations and the requirement $M_i/S \leq \rho$ is a sample approximation of a chance constraint and ρ is a desired probability level [28]. Specifically, when $\rho = 0$, no violations of the stabilizing constraint are allowed and, when $\rho = 1$, all stabilizing constraints are allowed to be violated. We then construct a reduced set of candidate policies $\bar{\mathbf{u}}_\mathbf{1} = \{\bar{u}^i_{0|t}, \ldots\} \subset \bar{\mathbf{u}}_\mathbf{0}$ that meet the constraint satisfaction criteria. To choose the best overall policy, the coordinate matrix V^{ec}_{set} is reduced into $\hat{V}^{ec}_{set}$ to account only for values that satisfy the constraint criteria. The policy $\bar{u}_{0|t}$ to be implemented is thus the one leading to the smallest cost measure (e.g., the expected value or median) among all columns in $\hat{V}^{ec}_{set}$. In this general case, we assume that at least one of the optimal inputs meets the constraint satisfaction criteria. The detailed approximation algorithm for this general case is outlined in Table 2. In the ideal case, this algorithm simply reduces to that in Table 1 because the matrices V^{ec}_{set} and $\hat{V}^{ec}_{set}$ coincide.

The proposed approach is an approximate strategy for SMPC but has several practical benefits. First of all, the approach is intuitive and easy to explain to industrial practitioners. Moreover, the approach is flexible in that it can handle different statistical measures and enables prioritization of constraints. A key observation and theoretical justification of the approach is that it *can do no worse than DMPC* in reducing the variances of state and manipulated input variables. This is because the control policy of DMPC is one of the candidate policies. Specifically, in DMPC, a representative value for the uncertainties (e.g., the mean or the worst-case value) is chosen to compute the control policy. The proposed approach can thus be interpreted as a strategy that seeks to improve on the DMPC policy or as a strategy that seeks to find the DMPC policy that best approximates the stochastic policy.

4. Application to Mechanical Pulping Processes

In this section, we describe an application of the proposed SMPC framework to a mechanical pulping process. We discuss the process and variables involved and provide numerical simulations to illustrate the effectiveness of the proposed approach.

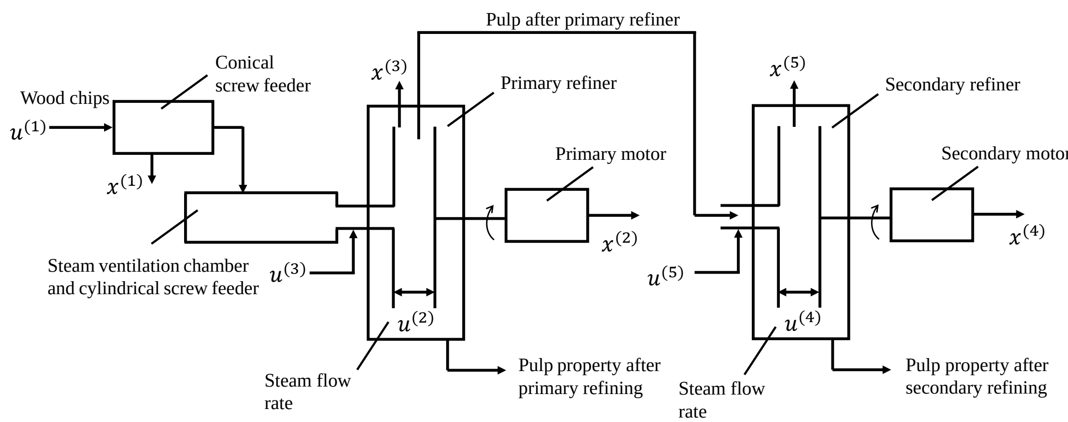

Figure 1: Schematic representation for two-stage HC refining process

4.1. Process Description

A two-stage HC MP process generally consists of three equipment units: wood chip pretreatment, wood chip refining, and pulp refining. The wood chips are introduced into the pretreatment unit where they are washed and screened to remove oversized or undersized chips and contaminants such as sands and stones. The wood chips are then steamed and preheated at atmospheric pressure around 100°C for the preparation of the refining process at the next stage. Wood chip refining, which aims at breaking wood chips into wood fibers, plays a major role in achieving the final pulp properties. There are normally two HC refiners, the primary HC refiner and the secondary HC refiner, as shown in Figure 1. The wood chips are introduced into the inlet of the primary HC refiner by a cylindrical chip transfer screw feeder that is manipulated to control the production rate. Then the wood chips are broken down into wood fibers as they pass through the two rotating discs of the refiner. Dilution water is fed into refiners to control the consistencies of the wood pulp in the refining zone. The secondary refiner is used to improve the pulp properties. At the pulp refining stage, the wood pulp is further processed to have the required properties. Table 3 lists key manipulated variables (MVs) and state variables (SVs) used to develop a discrete-time nonlinear model for the process. Another key variable is the specific energy (in MW/tonnes/day), which quantifies the energy consumed per tone of the dry pulp [23]. The specific energy has a significant effect on the pulp properties and is a key variable in advanced controller design for MP processes [19, 20, 29]. For the two-stage HC MP process, the total specific energy (TSE) is defined as the ratio of the total motor load and the production rate. In this work, the TSE is embedded directly in the cost function of the controller and this is used as an indicator of economic performance.

4.2. Dynamic Process Model

Modeling the two-stage HC MP process is challenging due to unknown mechanisms inside the pulp refiner as well as the inherently complex interactions among variables. The models of the process employed in this paper are based on our previous work [23, 26]. The nonlinear two-stage HC MP process can be written as:

$$x_{t+1}^{(1)} = a_1 x_t^{(1)} + b_1 k_a \cdot k_p \cdot s_c \cdot d_c \cdot u_t^{(1)}, \tag{9a}$$

$$x_{t+1}^{(2)} = a_2 x_t^{(2)} + b_2 \frac{k_{m_1} \cdot x_t^{(1)}}{u_t^{(3)}} (1 - e^{-10 u_t^{(2)}})(c_1 - e_1 \cdot u_t^{(3)}), \tag{9b}$$

$$x_{t+1}^{(3)} = a_3 x_t^{(3)} + b_3 \frac{100 x_t^{(1)}}{x_t^{(1)} + k_a \cdot u_t^{(3)} - k_{e_p} \cdot x_t^{(2)}}, \tag{9c}$$

$$x_{t+1}^{(4)} = a_4 x_t^{(4)} + b_4 \frac{k_{m_2} \cdot x_t^{(1)}}{u_t^{(5)}} (1 - e^{-10 u_t^{(4)}})(c_2 - e_2 \cdot u_t^{(4)}), \tag{9d}$$

$$x_{t+1}^{(5)} = a_5 x_t^{(5)} + b_5 \frac{100 x_t^{(1)}}{x_t^{(1)}/(0.01 x_t^{(3)}) + k_a \cdot u_t^{(5)} - k_{e_s} \cdot x_t^{(4)}}, \tag{9e}$$

where $x_t^{(i)}$, $u_t^{(i)}$, $i = 1, \ldots, 5$, are the i-th state and manipulated variables (defined in Table 3) at time t, respectively. a_i, $b_i = 1 - a_i$, $i = 1, \ldots, 5$, are process parameters and vary with the different refiners in each pulp mill. $s_c(\%)$ is the chip solid content. d_c (kg/m^3) is the chip bulk density. k_a, and k_p (m^3/rev) are constant parameters. k_{m_i}, k_{e_i}, c_i, and e_i, $i = 1, 2$, are the parameters for the i-th refiner which can be estimated from the industrial data and their values depend on the particular production lines. Moreover, since wood chips are the main raw materials in the two-stage HC MP process, variations in wood chips comprise the main disturbance that affect the final pulp properties. In this paper, the variations in wood chips, such as the bulk density (d_c) and the solid content (s_c) are considered as the random disturbances. The cost function of the controller (the TSE) is given by:

$$L^{ec}(z_t, v_t) = (y_t^{(2)} + y_t^{(4)})/y_t^{(1)} \tag{10}$$

where the dependence on z_t and v_t is implicit in the outputs y_t.

4.3. Simulation Results

4.3.1. Simulation I

We now demonstrate the effectiveness of the proposed algorithm. The process model is given in (9a)–(9e). The state and manipulated variables are listed in Table 3. The process dynamics are modeled through system identification with real industrial data. In the simulations, we assume that all the state variables are directly measurable and affected by random measurement noise (denoted as η_t). We also introduce random disturbances to the properties of the wood chips (denoted as ζ_t). To enforce stability, we obtain a factor ϵ_t by solving a reference tracking problem, as reported in [23].

Variations in the raw materials such as the chip bulk density and chip solid content are considered as random disturbances. We assume that disturbances $\{\zeta_t, \eta_t\}$ are normally distributed with zero mean and constant covariance:

$$\zeta_t \sim \mathcal{N}(0, Q_\zeta), \text{ and } \eta_t \sim \mathcal{N}(0, Q_\eta).$$

The prediction horizon is set to be $N = 30$. The sampling interval is chosen as 8 sec, and the simulation time is 120 sec. The other parameters used in the simulation are shown in Table 4. To demonstrate the effect of the number of scenarios S on control performance, we selected three cases with $S = 1, 5, 30$ and the SMPC controller is designed to minimize the median of the economic cost. When $S = 1$, it is assumed that only one sample is drawn (this approach is equivalent to a DMPC policy).

Table 5 shows the variances for states, manipulated inputs, and specific energy over time. The variance is used as a subject of volatility in the controller performance. It can be observed that by increasing S from 1 to 30, the variance of measured output variables, manipulated input variables and the specific energy have been reduced dramatically. This is because using a single scenario makes the control policy susceptible to variations in the disturbances while increasing the number of scenarios protects the controllers.

The simulation results are shown in Figures 2 3. Figures 2 and 3 illustrate the tracking performance of measured outputs and manipulated input variables. It can be observed that, in all cases, the controller is able to stabilize the system at the desired target steady-state. For $S = 1$, the controller gets close to instability at the beginning of the transition while, as S increases, the controller has better performance.

4.3.2. Simulation II

In this simulation example, we compare the benefits of the proposed modified SMPC (with scenario number $S = 5$) approach over traditional SMPC and DMPC from both the computational time and control performances aspects.

The proposed modified SMPC is computational efficient comparing with the classical SMPC. Take the SMPC with scenario number $S = 5$ as an example. For each scenario, the SMPC problem has a total of 300 variables and 610 constraints. The simulations were run on a 2-core computer with Intel Core i7-3537U CPU 2.00 GHz processors, and 8 GB RAM. We implement the two-stage HC MP process in `AMPL` and solved the nonlinear optimization problems using `Ipopt` [30]. Each MPC instance takes about 0.5s for the traditional DMPC (which is equivalent to the SMPC with scenario number $S = 1$), 4s for the proposed modified SMPC, and about 17s for classical SMPC to solve. The closed-loop simulation requires approximately 1 minute for the DMPC, 8.5 minutes for the modified SMPC, and 35 minutes for the traditional SMPC. The computational time is spent by the operating system comprises forecasting, optimization solution, and feasibility check.

We further compare the control performance of proposed modified SMPC approach with the scenario number $S = 5$ over DMPC and the traditional SMPC. The parameters used in this simulation are the same as we used in Simulation 4.3.1 and are listed in Table 4. For DMPC in this simulation, at each sampling time, we compute the optimal control policy and value function for every sampled scenario as a deterministic optimization problem. To make it a fair comparison with the proposed SMPC, the control policy for the DMPC is selected as the one that corresponds to the median of optimal value functions. Then the calculate control inputs by DMPC, the traditional SMPC, and the proposed SMPC are applied to the two-stage HC MP process separately with the same plant model, disturbances and measurement noises. The simulation results are shown in Figures 4 – 7. Figures 4 – 5 demonstrate the control performance of measured outputs and manipulated inputs, respectively. It can be found that the variances of measured output and manipulated input variables can be significantly reduced by the proposed SMPC comparing with DMPC. This observation

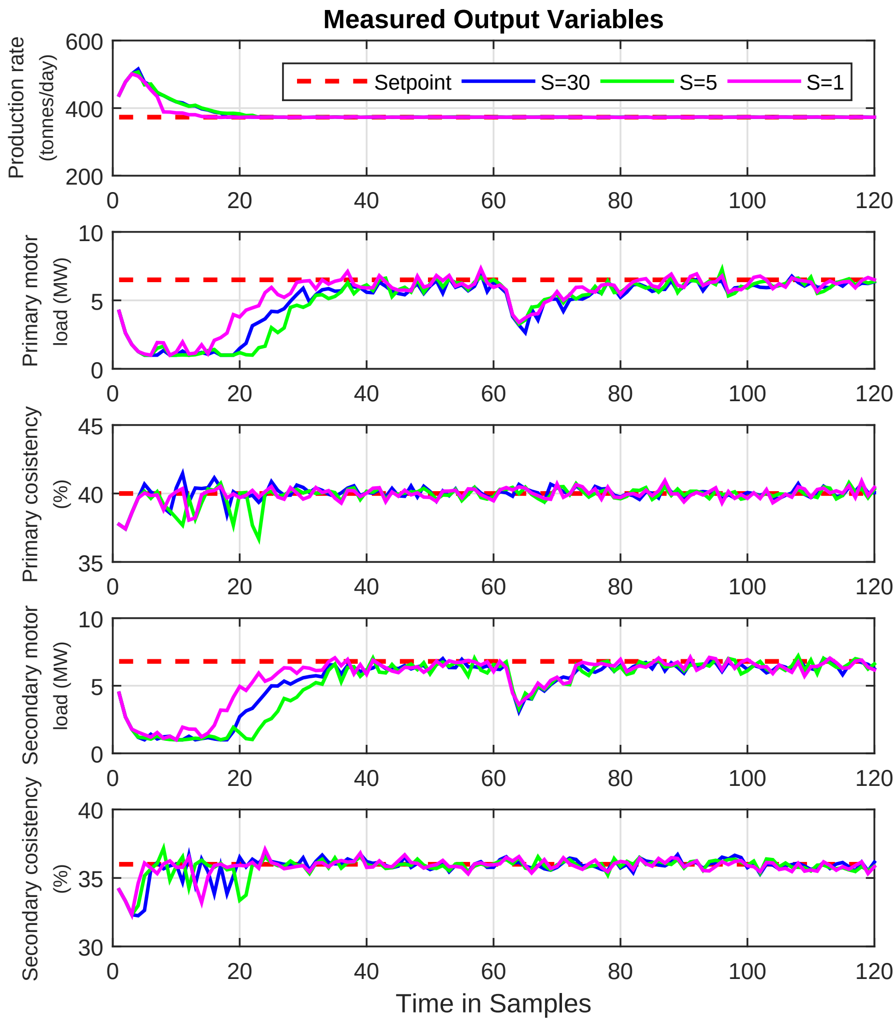

Figure 2: Closed-loop output policies obtained with the proposed controller with scenario number $S = 1, 5, 30$.

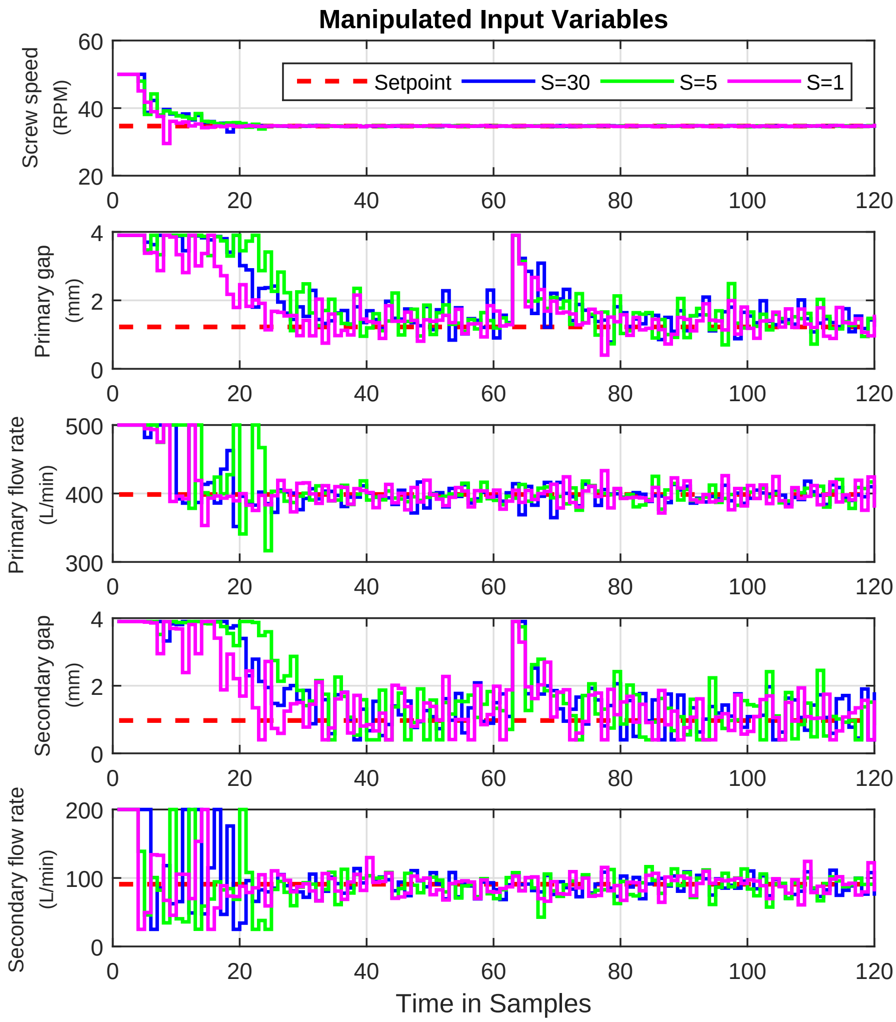

Figure 3: Closed-loop input policies obtained with the proposed controller with scenario number $S = 1, 5, 30$.

can be more clearly verified in Table 6. Moreover, the controller in DMPC is more likely be instability and drives the states beyond bound constraints (see the primary and secondary motor load plots in Figure 4). It also can be easily found from the simulation results that the proposed SMPC has only a slight performance penalty compared to traditional SMPC and the approximation algorithm is no worse than DMPC in terms of reducing the variances of state and manipulated input variables. The value functions of the two-stage HC MP process by using DMPC, traditional SMPC, and the proposed SMPC in Figure 6. It can be observed from Figure 6 that the calculated value function by DMPC is far off from the true value function. However, the calculated value function by using the modified SMPC is very close to actual value function. In another word, the modified SMPC can provide a reliable prediction of the future energy consumption. The specific energy consumptions by the two-stage HC MP process using different MPC schemes are compared in Figure 7 in which it shows that the MP process consumes the least amount of the specific energy using the proposed SMPC.

5. Conclusions

The main contribution of this paper is to develop an approach to obtain approximate control policies for stochastic MPC. This approach seeks to address computational tractability issues of stochastic MPC and offers flexibility to handle diverse statistical measures. The proposed approach offers the guarantee that it can do no worse than DMPC in decreasing the variances of the state and manipulated input variables and can be interpreted as a strategy that seeks to find a deterministic policy that gives the best approximation to a SMPC policy. A simulation example for a two-stage HC MP process demonstrates the effectiveness of the proposed approach.

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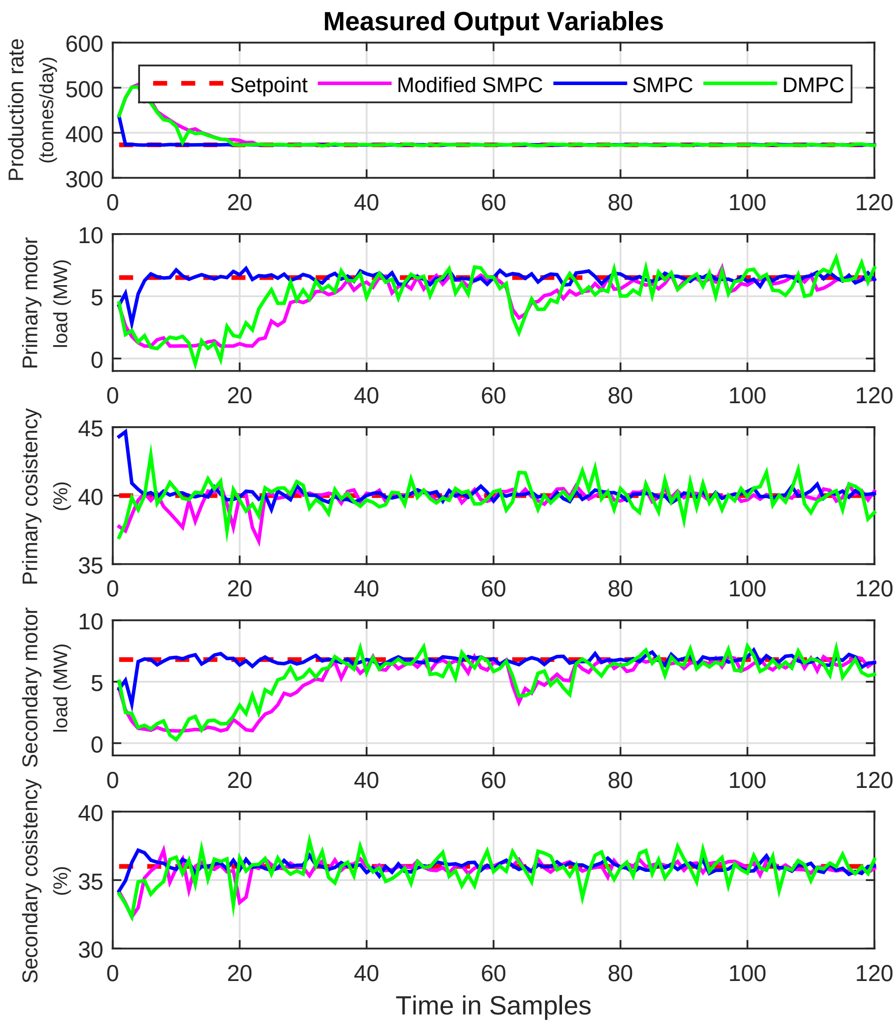

Figure 4: Closed-loop output policies obtained with the proposed controller with scenario number $S = 30$ and DMPC.

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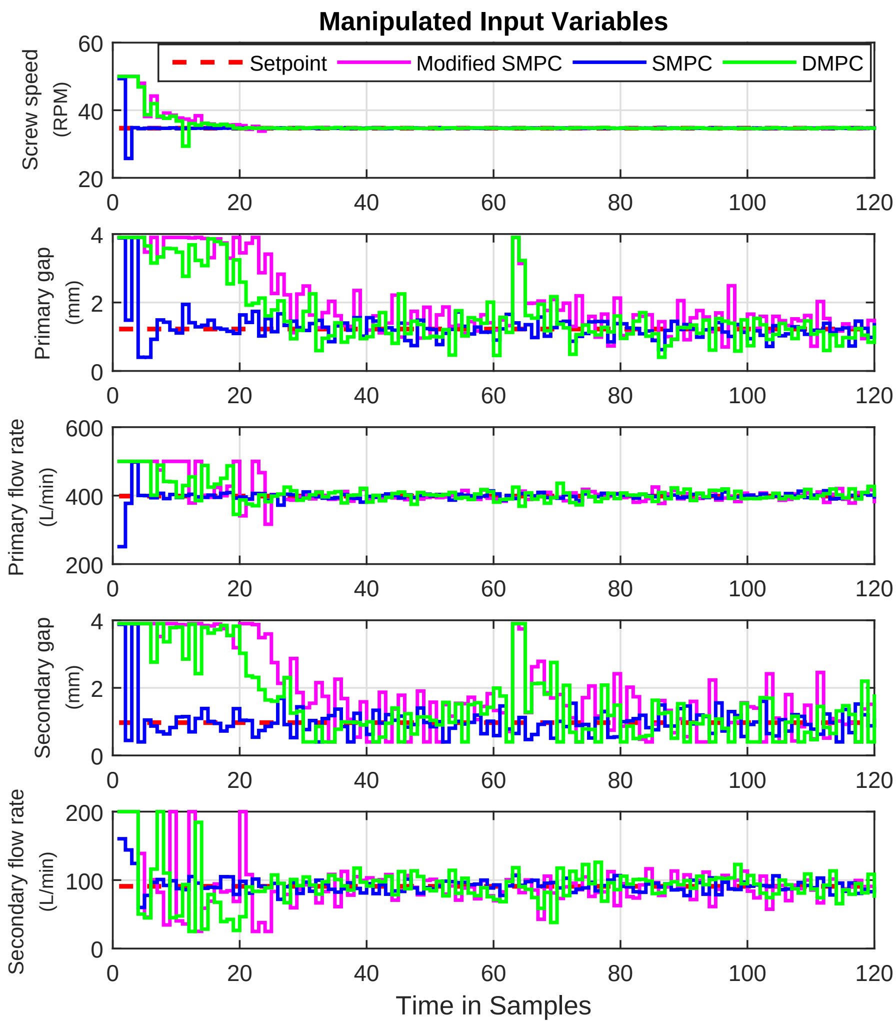

Figure 5: Closed-loop input policies obtained with the proposed controller with scenario number $S = 30$ and DMPC.

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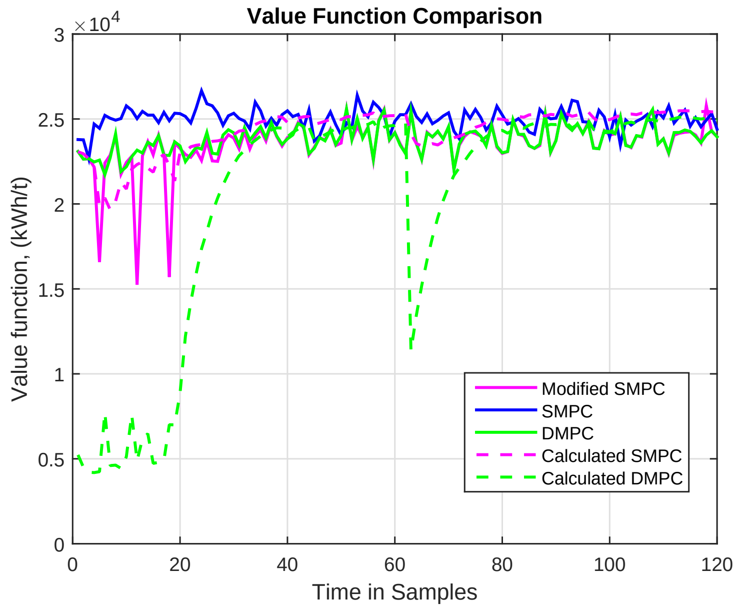

Figure 6: Comparison of value functions with the proposed controller $S = 1, 30$ and DMPC.

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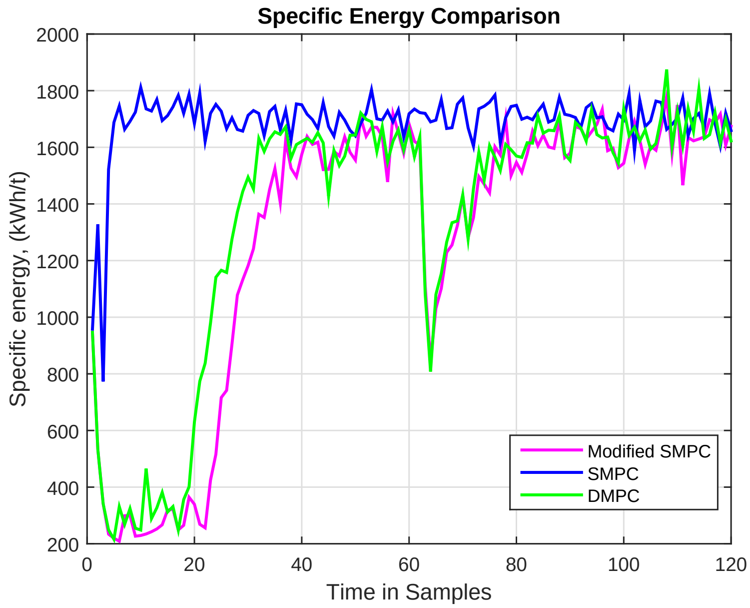

Figure 7: Comparison of energy reduction achieved with the proposed controller DMPC.

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Table 2: Implementation of proposed approximate SMPC scheme in general case

Algorithm for general cases

Input: $x_0 \in \mathbb{X}$, $\rho \in [0,1)$, set $t \leftarrow 0$ and $\epsilon_0 \leftarrow +\infty$.

For $t = 0, \ldots,$ simulation duration **do**

1: Draw S scenarios of $\{\zeta^i_{k|t}, \eta^i_{k|t}\}_{k=0}^{N-1}$, $i = 1, \ldots, S$.

2: **for** $i = 1, \ldots, S$, **do**

2.1. Compute optimal input $\bar{u}^i_{0|t}$ and its corresponding optimal value function $\bar{V}^{ec}_t(\bar{u}^i_{0|t}, \{\zeta^i_{k|t}, \eta^i_{k|t}\})$ for the i-th scenario.

2.2. Given $\bar{u}^i_{0|t}$ computed in step 2.1, evaluate $V^{ec}_t(\bar{u}^i_{0|t}, \{\zeta^j_{k|t}, \eta^j_{k|t}\})$ for the rest scenarios $\{\zeta^j_{k|t}, \eta^j_{k|t}\}_{k=0}^{N-1}$, $\forall j = 1, \ldots, S$, and $j \neq i$.

end for

3: Combining the results computed in step 2, construct the optimal input set $\mathbf{\bar{u}_0} = \{\bar{u}^1_{0|t}, \cdots, \bar{u}^S_{0|t}\}$ and the coordinate matrix V^{ec}_{set} as in (8).

4: Check the constraints (7e) and (7f) and count the number of the violations M_i and N_i, $i = 1, \cdots, S$, for each sampled scenarios $\{\zeta^j_{k|t}, \eta^j_{k|t}\}_{k=0}^{N-1}$, $j = 1, \cdots, S$, under the optimal manipulated input $\bar{u}^i_{0|t}$, $i \neq j$.

5: Find the inputs $\bar{u}^i_{0|t} \in \mathbf{\bar{u}_0}$ satisfying the conditions: (a) $M_i/S \leq \rho$; (b) have the least number of N_i, $i \in \{1, \cdots, S\}$. Stack the qualified optimal inputs into a set $\mathbf{\bar{u}_1} = \{\bar{u}^i_{0|t}, \cdots\} \subset \mathbf{\bar{u}_0}$

6: Choose the best overall optimal input as follows:

6.1. **If** there is only one candidate input in $\mathbf{\bar{u}_1}$,

then the best overall optimal input is chosen to be this candidate input $\bar{u}^l_{0|t}$.

6.2. **If** there are more than one candidate inputs in $\mathbf{\bar{u}_1}$

then

6.2.1. A new matrix $\hat{V}^{ec}_{set}$ is formed with value functions for each scenario under the qualified optimal inputs in $\mathbf{\bar{u}_1}$;

6.2.2. The best optimal input $\bar{u}^l_{0|t}$ corresponds to the row that contains the minimal measure among all rows in the new matrix $\hat{V}^{ec}_{set}$.

7: Set $u_t \leftarrow \bar{u}^l_{0|t}$. Implement u_t to the plant and obtain the state variables x_{t+1}.

8: Set $\epsilon_{t+1} < \epsilon_t$.

End

Table 3: List of process variables for the two-stage HC MP process

MVs	Name (unit)	Notation	SVs	Name (unit)	Notation
$u^{(1)}$	Chip transfer screw speed (rpm)	R	$x^{(1)}$	Production rate (tonnes/day)	P
$u^{(2)}$	Primary refiner plate gap (mm)	G_p	$x^{(2)}$	Primary motor load (MW)	M_p
$u^{(3)}$	Primary dilution flow rate (kg/s)	D_p	$x^{(3)}$	Primary consistency (%)	C_p
$u^{(4)}$	Secondary refiner plate gap (mm)	G_s	$x^{(4)}$	Secondary motor load (MW)	M_s
$u^{(5)}$	Secondary dilution flow rate (kg/s)	D_s	$x^{(5)}$	Secondary consistency (%)	C_s

Table 4: Simulation parameters for the SMPC controller

Symbol	Values	Description
T	120s	Simulation length
N	30	Prediction horizon for the *m-econ* MPC controller
S	{1, 5, 30}	Number of the scenarios
Q_w	0.25	Variance of the disturbance/model uncertainty
Q_v	0.1	Variance of the measurement
ρ	0.2	Violation tolerance for stabilizing constraint

Table 5: Variances for outputs, inputs, and specific energy for scenario cases $S = 1, 5, 30$

	Variance of output variables
$S = 1$	[0.1259, 0.5238, 0.1302, 0.4630, 0.0957]
$S = 5$	[0.0958, 0.3805, 0.0754, 0.3911, 0.0651]
$S = 30$	[0.0492, 0.3173, 0.0575, 0.2625, 0.0525]
	Variance of input variables
$S = 1$	[0.0069, 0.2344, 199.1535, 0.4471, 199.5652]
$S = 5$	[0.0065, 0.1995, 93.1120, 0.4269, 164.2665]
$S = 30$	[0.0020, 0.1530, 69.4539, 0.2386, 84.6686]
	Variance of specific energy
$S = 1$	1.8430×10^6
$S = 5$	5.6068×10^5
$S = 30$	5.4775×10^5

Table 6: Variances for outputs, inputs, and specific energy comparison between the proposed SMPC, traditional SMPC, and DMPC

	Variance of output variables
DMPC	[0.6018, 0.8008, 0.6071, 0.9440, 0.6145]
Traditional SMPC	[0.0738, 0.0864, 0.0647, 0.0676, 0.0771]
Modified SMPC, $S = 5$	[0.0492, 0.3173, 0.0575, 0.2625, 0.0525]
	Variance of input variables
DMPC	[0.0061, 0.2752, 171.7846, 0.5106, 242.5241]
Traditional SMPC	[0.0024, 0.0484, 35.1460, 0.0876, 51.7882]
Modified SMPC, $S = 5$	[0.0020, 0.1530, 69.4539, 0.2386, 84.6686]
	Variance of specific energy
DMPC	5.2108×10^5
Traditional SMPC	3.2542×10^5
Modified SMPC, $S = 5$	5.4775×10^5