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Deep Neural Network Approximation of Nonlinear Model Predictive Control

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Model Predictive Control

Nonlinear Model Predictive control

- Repeatedly solve the optimal control problems online with different x_0

$$\begin{aligned} \min_{x(k),u(k)} \quad & \sum_{k\in\mathcal{T}} l(x(k),u(k)) + V_f(x(N)) \\ \text{s.t.} \quad & x(k+1) = f(x(k),u(k)) \\ & x(0) = x_0 \\ & x(k) \in \mathbb{X}, x(N) \in \mathbb{X}_f, u(k) \in \mathbb{U} \\ & \forall k \in \mathcal{T} \end{aligned}$$

- Implicit control law $u(0)=\kappa_N(x_0)$, expensive online computational cost

Explicit MPC

- Compute the optimal control law offline as a function of all possible states
- Multi-Parametric optimization
- Negligible online, intractable offline for process with 10+ variables

Deep Learning Based Model Predictive Control

Two-step / "Optimize-then-train" approach

(Zoppoli 1995, Lantos 2015, Zhanfg 2018, Lucia 2018, Gopaluni 2018)

- **Optimize** control actions for multiple initial states to obtain $(x_{0,s}, \kappa_N(x_{0,s}))$
- **Train** the neural network to obtain the control law $\hat{u}(0) = \hat{\kappa}_N(\pi, x_0)$

Advantages

- Negligible online computational cost

Disadvantages

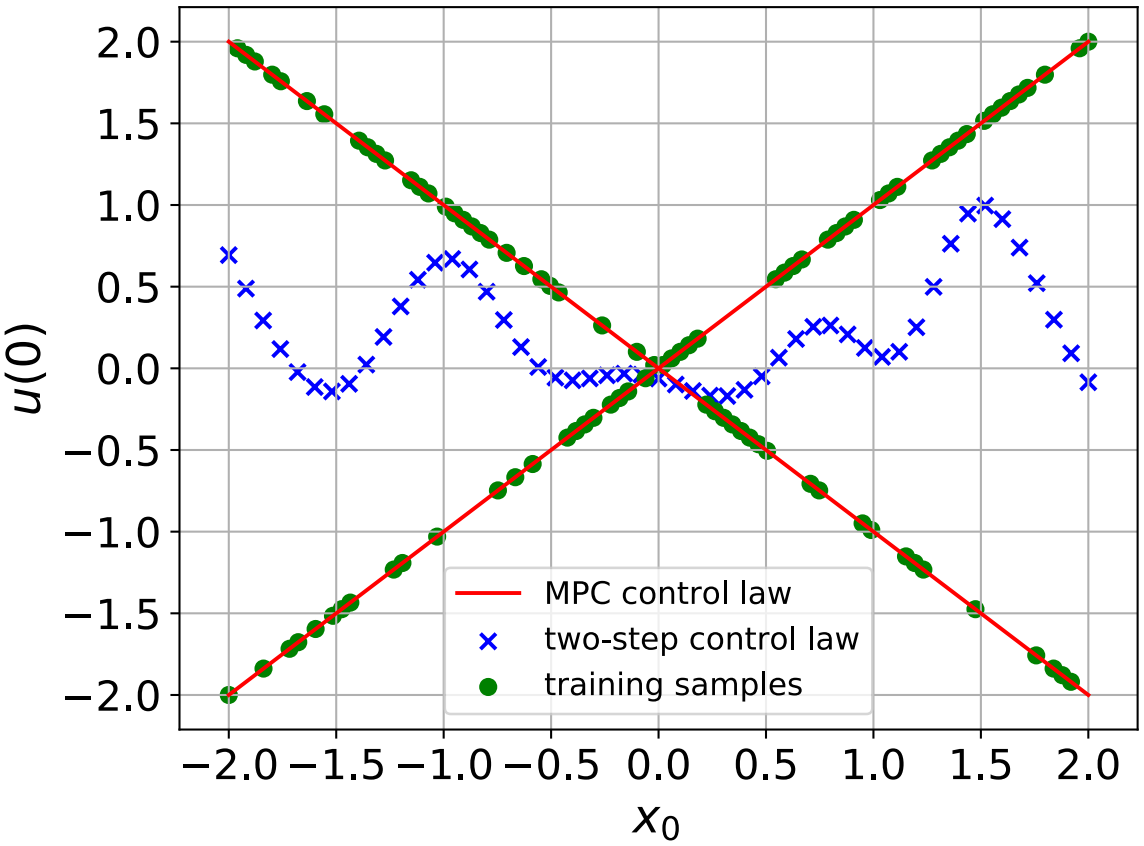
- Training error of the NN can lead to sub-optimal or even infeasible action even for training samples
- Those errors would accumulate through time (poor closed loop performance)
- Multiple optimal control actions for the same initial states
- Multiple local optimal control actions

Illustrative Example

$$\min_{x(k),u(k)} \quad \sum_{k=0}^{1} x(k)^2$$

$$\text{s.t.} \quad x(1) = x(0)^2 - u(0)^2$$

$$x(0) = x_0$$



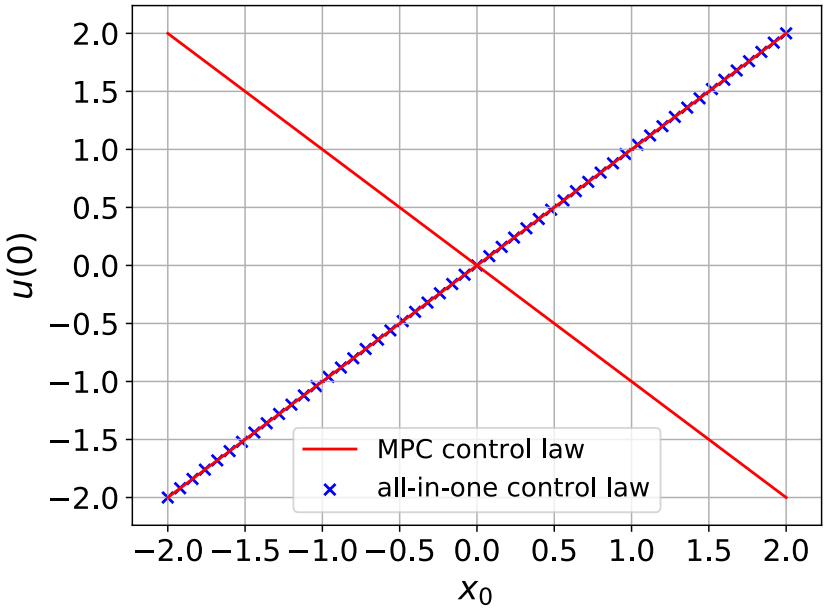
All-in-one/“Optimize-and-train” Approach: Stochastic Optimization

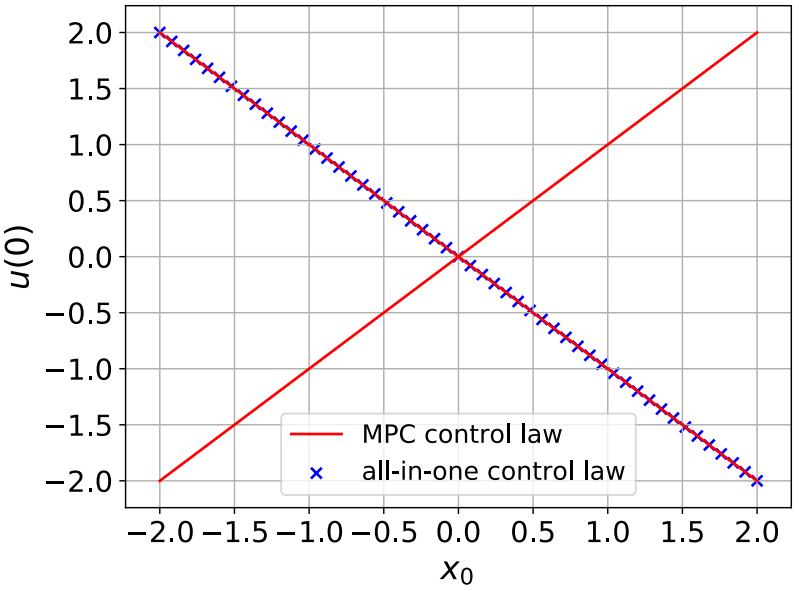
$$
\begin{aligned}
\min_{\pi, x_s(k), u_s(k)} \quad & \sum_{s \in \mathcal{S}} \sum_{k \in \mathcal{T}} l(x_s(k), u_s(k)) + V_f(x_s(N)) \\
\text{s.t.} \quad & x_s(k+1) = f(x_s(k), u_s(k)) \\
& u_s(k) = \hat{\kappa}_N(\pi, x_s(k)) \\
& x_s(0) = x_{0,s} \\
& x_s(k) \in \mathbb{X}, x_s(N) \in \mathbb{X}_f, u_s(k) \in \mathbb{U} \\
& \forall s \in \mathcal{S}, \forall k \in \mathcal{T}
\end{aligned}
$$

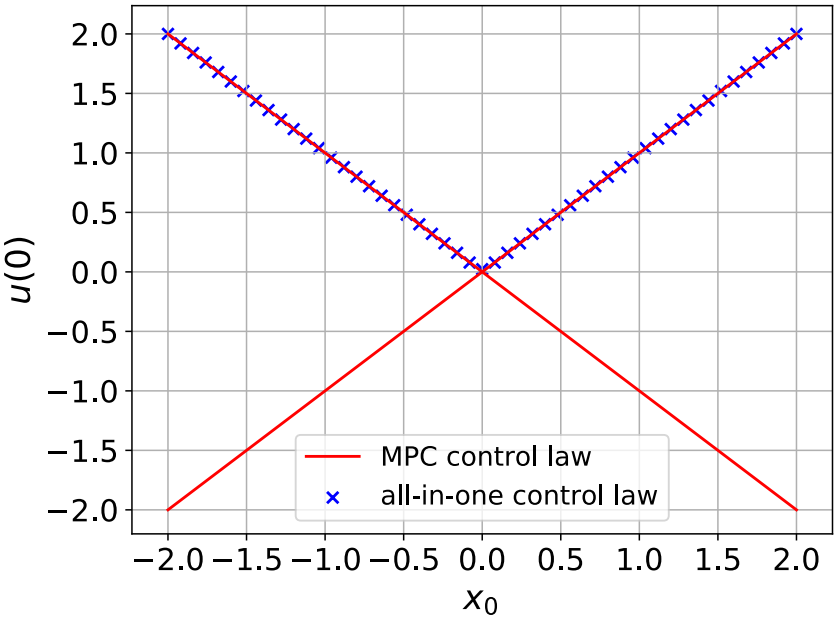
All-in-one/“Optimize-and-train” approach

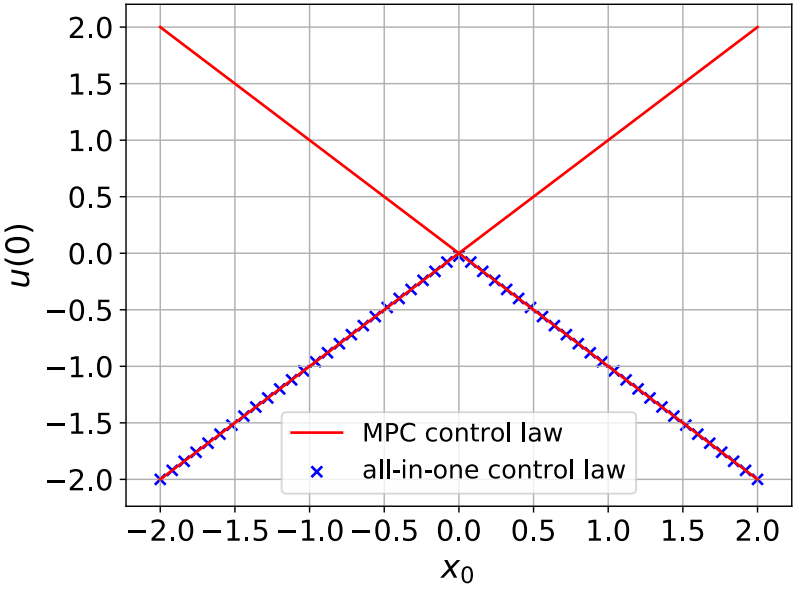
- Solve only one large scale optimization problem
- Decide the control law directly instead of control actions
- Optimize closed loop performance directly
- Constraints are satisfied at least for training samples
- Parallel solvers (e.g. PIPS-NLP) can exploit the structure on HPC (CPUs)
- Links to policy search/reinforcement learning

Illustrative Example









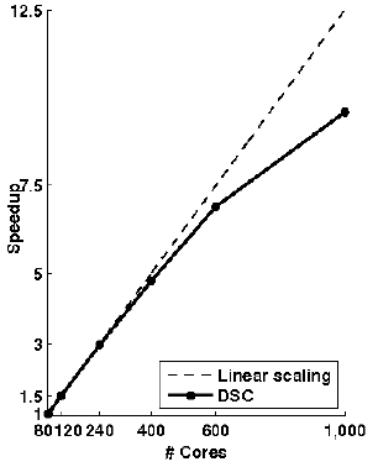
Structured Nonlinear Optimization (Scalable Linear Algebra)

$$\min_{\pi, x_s} \sum_{s \in \mathcal{S}} f_s(\pi, x_s)$$
$$\text{s.t.} \quad c_s(\pi, x_s) \geq 0, \quad s \in \mathcal{S}$$
$$x_s \in X_s, \quad s \in \mathcal{S}$$

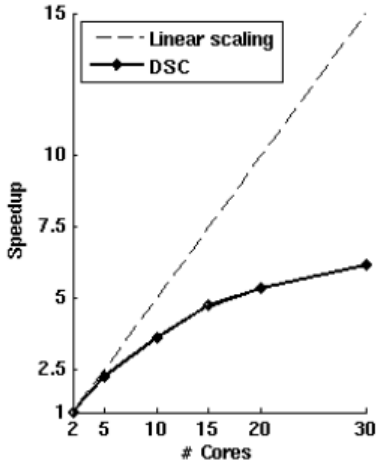
Argonne's Fusion Cluster
320 Nodes
12.5 TB Memory

Structured Linear System

$$\begin{bmatrix} K_1 & & & B_1 \\ & K_2 & & B_2 \\ & & \ddots & \vdots \\ & & K_S & B_S \\ B_1^T & B_2^T & \dots & B_S^T & K_0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_S \\ q_0 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_S \\ r_0 \end{bmatrix}$$



Loose Coupling



Tight Coupling

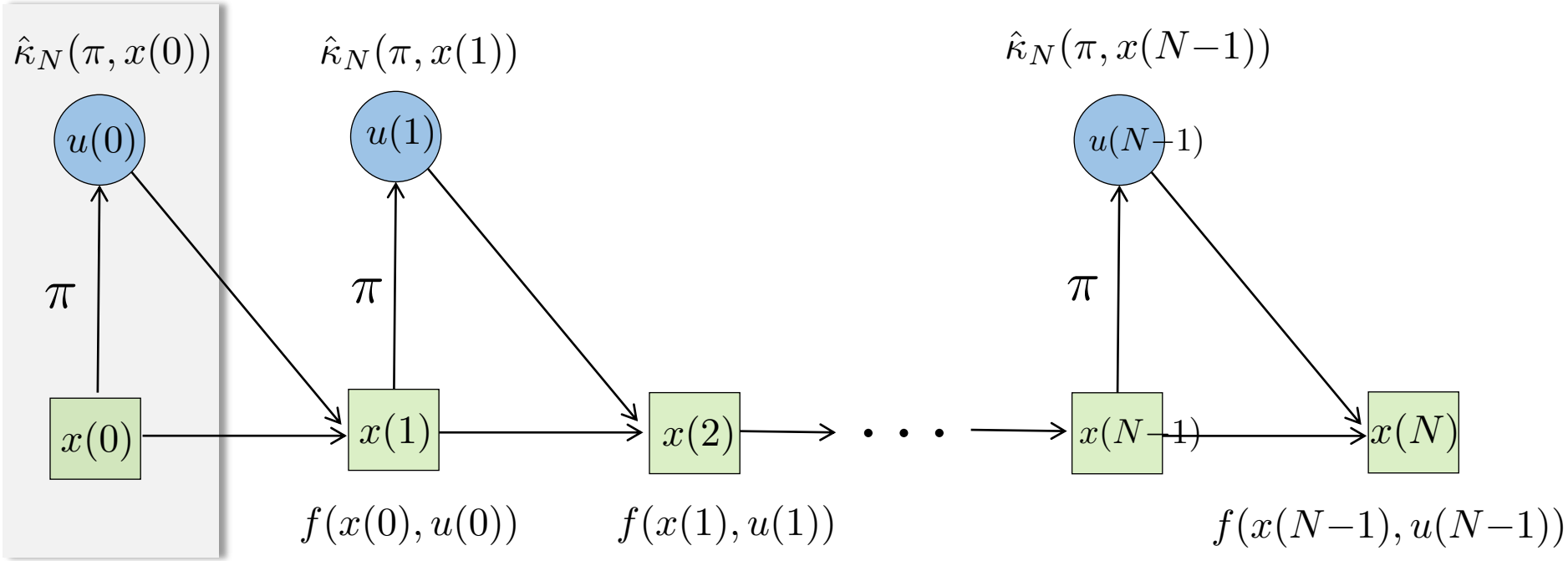
Schur Decomposition

$$\left(K_0 - \sum_{s \in \mathcal{S}} B_s^T K_s^{-1} B_s \right) q_0 = r_0 - \sum_{s \in \mathcal{S}} B_s^T K_s^{-1} r_s$$

$$K_s q_s = r_s - B_s q_0, \quad s \in \mathcal{S}$$

Cao et al. "Scalable modeling and solution of stochastic multi objective optimization problems." Computers & Chemical Engineering 99 (2017): 185-197.

“Optimize-and-train”/ All-in-one Approach: Recurrent Neural Network



The stochastic optimization problem can be reformulated as the training of RNN

- Existing packages (e.g. TensorFlow, Flux) can exploit the structure on GPUs
- Simple Input constraints (bounds) can always be satisfied by the design of NN
- Treat state constraints as soft constraints

Implementation

```
function step_model(x_k, u_k)
    return x_k.^2 - u_k.^2
end
control_law = Chain(
  Dense(1, 20, σ),
  Dense(20, 1))
function loss(x0, setpoint)
    x_k = x0
    x = nothing
    for t = 1:N
        u_k = control_law(x_k)
        x_k = step_model(x_k, u_k)
        x = t==1 ? x_k : vcat(x,x_k)
    end
    return Flux.mse(x,setpoint)
end
data=[([I],zeros(N)) for I in -2:0.1:2]
opt = ADAM()
Flux.@epochs 1000 Flux.train!(loss, Flux.params(control_law), data,
    opt)
```

Additional 10 lines to implement input/state constraints

Feasibility

Constraint Violation

For state constraints:

$g(x) \leq 0$

$$
\begin{aligned}
C_v(\pi) = \max_{x(0)\in\mathbb{X}_0, x(k), u(k)} \quad & \|[g(x)]_+\| \\
\text{s.t.} \quad & x(k+1) = f(x(k), u(k)) \\
& u(k) = \hat{\kappa}_N(\pi, x(k)) \\
& \forall k \in \mathcal{T}
\end{aligned}
$$

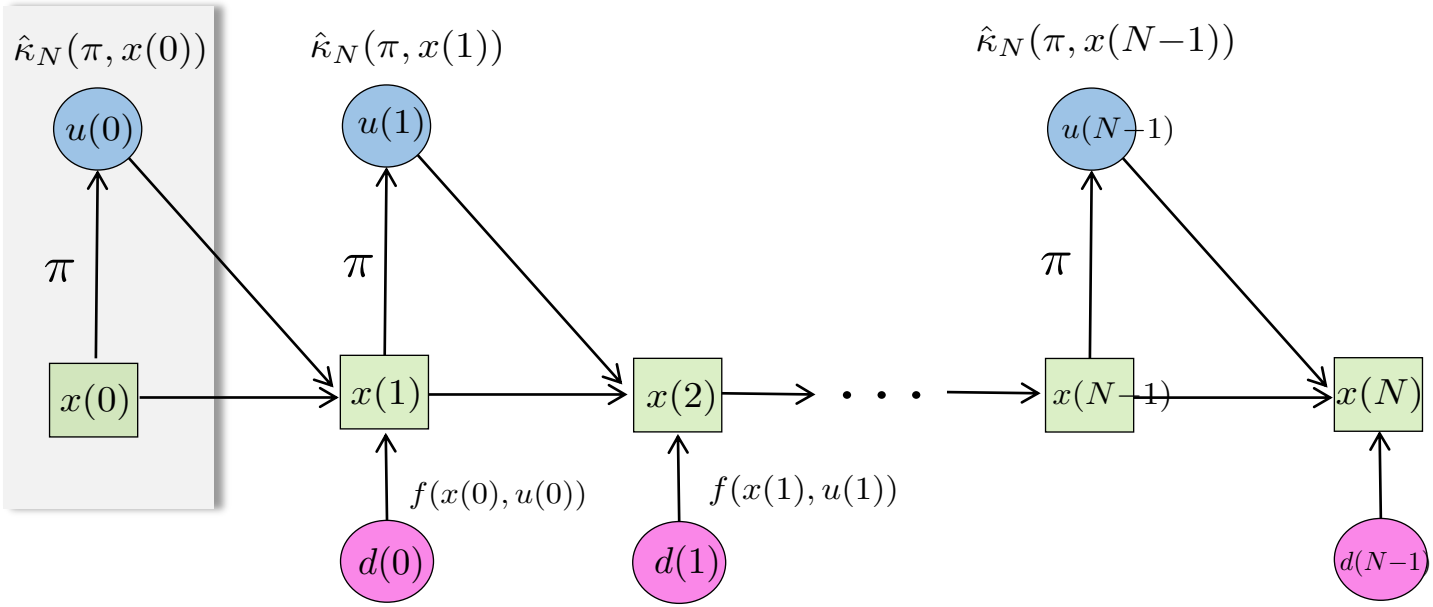
If $C_v(\pi) > 0$

- Select the scenario leading to the largest violations
- Add the scenario to the training scenarios, and optimize the control law again
- Or chose tighter state constraints in the optimization/training

If $C_v(\pi) = 0$

- It means the control law can drive any state $x_0 \in \mathbb{X}_0$ to $\mathbb{X}_f$ in N steps
- Assume a local control law $\mu_f(x_0) = Kx_0$ can stabilize any $x_0 \in \mathbb{X}_f$, then DNN control law is stable
- Or we can train the control law to ensure that states in $\mathbb{X}_f$ remain in the zone

Uncertainty



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\text{s.t.} \quad & x_s(k+1) = f(x_s(k), u_s(k), d_s(k)) \\
& u_s(k) = \hat{\kappa}_N(\pi, x_s(k)) \\
& x_s(0) = x_{0,s} \\
& x_s(k) \in \mathbb{X}, x_s(N) \in \mathbb{X}_f, u_s(k) \in \mathbb{U} \\
& \forall s \in \mathcal{S}, \forall k \in \mathcal{T}
\end{aligned}
$$

Each scenario has data $[x_{0,s}, d_s(k)]$

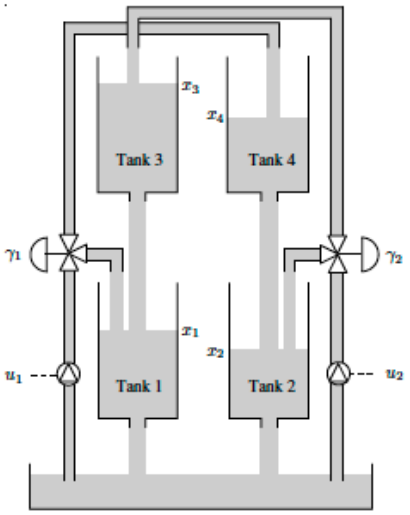
Nonlinear Quadtank Problem

$$\frac{dz_1}{dt} = -\frac{a_1}{A_1}\sqrt{2g(z_1 + x_{1s})} + \frac{a_3}{A_1}\sqrt{2g(z_3 + x_{3s})} + \frac{\gamma_1}{A_1}(v_1 + u_{1s})$$

$$\frac{dz_2}{dt} = -\frac{a_2}{A_2}\sqrt{2g(z_2 + x_{2s})} + \frac{a_4}{A_2}\sqrt{2g(z_4 + x_{4s})} + \frac{\gamma_2}{A_2}(v_2 + u_{2s})$$

$$\frac{dz_3}{dt} = -\frac{a_3}{A_3}\sqrt{2g(z_3 + x_{3s})} + \frac{(1-\gamma_2)}{A_3}(v_2 + u_{2s})$$

$$\frac{dz_4}{dt} = -\frac{a_4}{A_4}\sqrt{2g(z_4 + x_{4s})} + \frac{(1-\gamma_1)}{A_4}(v_1 + u_{1s})$$



Training scenarios:

- 81 initial state scenarios with each state variable discretized by 3 points
- for ideal NMPC, we need to solve 81 *20 = 1620 optimization problems (N=20 steps)
- These 1620 data pairs are used in the two-step approach

Test scenarios:

- 256 initial state scenarios with each state variable discretized by 4 points

NN controller:

- Structure 4 - 10 - 2 - 2
- Both hidden layers use the activation function tanh
- To guarantee the satisfaction of input constraints, the last layer projects values in the range of [-1,1] to $[v_{min}, v_{max}]$

Nonlinear Quadtank Problem

Performance of different controllers

	training		testing	
	cost	Cons. Viol.	cost	Cons. Viol.
ideal NMPC	582.18	0	488.53	0
two-step	582.64	1.85	492.83	1.85
all-in-one	582.29	0	488.95	0.075

Averaged online and offline computational time

	online (s)	offline (s)
ideal NMPC	0.016	-
two-step	4e-5	2134
all-in-one	4e-5	1194

Performance of two-step method with different DNN layers

# of layers	training		testing	
	cost	Cons. Viol.	cost	Cons. Viol.
2	582.64	1.8502	492.83	1.8502
4	584.40	0.599	490.26	1.008
6	583.74	1.12	491.06	1.18
8	586.26	0.259	492.23	1.240
10	582.87	0.636	489.71	0.746
12	586.08	0.513	492.84	1.250

Summary

All-in-one Approach

- Still works even if optimal control actions are not unique
- Constraints are satisfied at least for training samples
- RNN reformulation might reduce the training time